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THE PEBBLING NUMBER OF ZIG-ZAG CHAIN GRAPH OF EVEN CYCLES

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Abstract

This paper considers the pebbling game on a zig-zag chain graph of even cycles. A pebbling move on a connected graph G consists of removing two pebbles from some vertex and adding one on an adjacent vertex. The pebbling number of a connected graph G, denoted by $f(G)$, such that any distribution of $f(G)$ pebbles on G allows one pebble to be moved to any specified but arbitrary vertex by a sequence of pebbling moves. In this paper we compute the pebbling number of zig-zag chain graphs of even cycles and show that they satisfy the two-pebbling property.

Keywords: Graph pebbling, Zig-zag chain graph, cycle.

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1. Introduction

All graphs in this paper are assumed to be simple, finite, and undirected. For graph theoretic terminologies we follow the notion of [1].

A recent development in graph theory, the pebbling game was first suggested by Lagarias and Saks as a tool for solving number-theoretical conjecture of Erdos. Chung used this tool to establish the results concerning pebbling number. In doing so she introduced the pebbling to the literature [2].

Consider a simple graph G with vertex set $V(G)$ and edge set $E(G)$. For a graph G, let D be a distribution of pebbles on the vertices of G. For any vertex v of G, p_v denotes the number of pebbles on v in D. For $K \subseteq V(G)$, we denote $p(K) = \sum_{v \in V(K)} p(v)$. A *pebbling move* consists of removing two pebbles from one vertex and then placing one pebble on an adjacent vertex. In this paper, the letter v will frequently be used to denote the specified vertex of the graph under consideration.

Definition 1.1. [2] The *pebbling number of a vertex* v in G is the smallest number $f(G,v)$ such that every placement of $f(G,v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves.

The *pebbling number* of G is the smallest number, $f(G)$, such that from any placement of $f(G)$ pebbles it is possible to move one pebble to any specified target vertex by a sequence of pebbling moves. Thus, $f(G)$ is the maximum value of $f(G,v)$ overall vertices v.

Implicit in this definition is the fact that if after moving to a specified vertex v one desires to move to another specified vertex, the pebbles reset to their original initial distribution. Now, we state some facts from [3] about $f(G)$.

1. For any vertex v of a graph $G, f(G,v) \geq n$ where $n = |V(G)|$.
2. The pebbling number of a graph G satisfies $f(G) \geq max\{2^{diam(G)}, |V(G)|\}$, where $diam(G)$ is the diameter of the graph G.

Chung [2] also defined the *two-pebbling property* of a graph.

Definition 1.2 Let D be a distribution of pebbles on G, let q be the number of vertices with at least one pebble. We say that G satisfies the $2-$ *pebbling property* if for any distribution with more than $2f(G)-q$ pebbles, it is possible to move two pebbles to any specified vertex.

This paper is organized as follows. In Section 2, we define the zig-zag chain graph of even cycles and give some preliminary results which are used in our main results. In Section 3, we state and prove our main results of pebbling on zig-zag chain graph of even cycles.

2. Preliminaries

We now define the zig-zag chain graph of even cycles and discuss some results which are useful for subsequent sections.

Definition 2.1. The *zig-zag chain graph of even cycles* denoted by ZZ_n, is a graph which consists of zig-zag sequence of n even cycles, C_{2k} with $k \geq 3$. We have the following vertex set and edge set of ZZ_n for n even as follows:

$V(ZZ_n) = \{a_i, b_i : 1 \leq i \leq n(k-1)\} \cup \{x, y\}$ and

$$E(ZZ_n) = \{a_i a_{i+1}, b_i b_{i+1} : 1 \leq i \leq n(k-1) - 1\} \cup \{xa_1, xb_1, ya_{n(k-1)}, yb_{n(k-1)}\} \cup \left\{a_{i(k+1)-1} b_{i(k+1)-2}, a_{j(k+1)} b_{j(k+1)+1} : 1 \leq i \leq \frac{n}{2}, 1 \leq j \leq \frac{n-2}{2}\right\}$$

For n odd, we have the following vertex set and edge set.

$V(ZZ_n) = \{a_i, b_i : 1 \leq i \leq n(k-1)\} \cup \{x, y\}$ and

$$E(ZZ_n) = \{a_i a_{i+1}, b_i b_{i+1} : 1 \leq i \leq n(k-1) - 1\} \cup \{xa_1, xb_1, ya_{n(k-1)}, yb_{n(k-1)}\} \cup \left\{a_{i(k+1)-1} b_{i(k+1)-2}, a_{j(k+1)} b_{j(k+1)+1} : 1 \leq i, j \leq \frac{n-1}{2}\right\}$$

The Figure 1.1 depicts the graph ZZ_3 for $k = 3$.

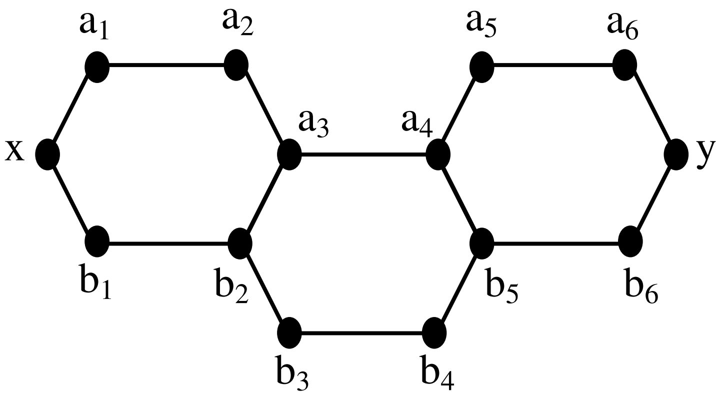

Figure 1.1.

The reader can easily view that ZZ_n has n copies of C_{2k}, and label each cycle as $C_1, C_2, \ldots,$ and C_n in order. Here, we present some results that will be used in the proof of main results.

Theorem 2.2. *[4] Let P_n be the path with n vertices. Then*

(1) $f(P_n) = 2^{n-1}$ *and*

(2) P_n *satisfies the* $2-$ *pebbling property.*

Theorem 2.3. *[4] Let C_n denote a simple cycle with n vertices, where $n \geq 3$. Then*

(1) $$f_t(C_n) = \begin{cases} 2^k, & n(=2k) \text{ is even} \\ 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1, & n(=2k+1) \text{ is odd} \end{cases}$$

(2) The graph C_n satisfies the $2-$ *pebbling property.*

Observation 2.4. *For any given integers m, s and t,*

we have $2^{(s+t)m} - 2^{tm} \geq 2^{(s+t-1)m}$, *where* $t \geq 2$ *and* $m, s \geq 1$.

Proof. We have to prove $2^{(s+t)m} - 2^{tm} - 2^{(s+t-1)m} \geq 0$.

Since $t \geq 2$, take $t = t_1 + 1$ and $t_1 \geq 1$.

We have $2^{(s+t_1+1)m} - 2^{(t_1+1)m} - 2^{(s+t_1)m}$

$$= 2^{(t_1+1)m}\left(2^{sm} - 1 - 2^{(s-1)m}\right)$$

$$= 2^{(t_1+1)m}2^{(s-1)m)}\left(2^m - \frac{1}{2^{(s-1)m)}} - 1\right)$$

$$\geq 0, \text{ since } m \geq 1 \text{ and } \left(2^m - \frac{1}{2^{(s-1)m)}} - 1\right).$$

3.Main Results

In this section, we find the pebbling number of the graph ZZ_n and show that ZZ_n satisfies both the 2−pebbling by using induction. Observation 2.4. is a useful tool for proving the main results slightly simple. It describes how many pebbles can be retained on the graph ZZ_n after using certain amount of pebbles by pebbling moves. Before we proceed, we give the following remarks.

Remark 3.1. Consider a graph G with *n* vertices and $f(G)$ pebbles on it and we choose a target vertex v from G. If $p_v = 1$ or $p_u \geq 2$, where $uv \in E(G)$, then we can move a pebble to v easily. So we always assume that $p_v = 0$ and $p_u \leq 1$ for all $uv \in E(G)$, when v is the target vertex.

Remark 3.2. Consider the graph G with n vertices and $2f(G) - q + 1$ pebbles on it and we choose a target vertex v

from G. If $p_v = 1$, then the number of pebbles remaining in G is $2f(G) - q \geq f(G)$, since $f(G) \geq n$ and $q \leq n$, and hence we can move the second pebble to v. Let us assume that $p_v = 0$. If $p_u \geq 2$ where $uv \in E(G)$, we move a pebble to v from u. Then the graph G has at least $2f(G) - q + 1 - 2$ pebbles, since $f(G) \geq n$ and $q \leq n - 1$, and hence we can move the second pebble to v. So, we always assume that $p_v = 0$ and $p_u \leq 1$ for all $uv \in E(G)$, when v is the target vertex.

Theorem 3.3. *In a zig-zag chain graph of even cycles C_{2k}, denoted by ZZ_n with a specified vertex v, the following are true for $k \geq 3$.*

1. *If $2^{n(k-1)+1}$ pebbles are assigned to the vertices of the graph ZZ_n, one pebble can be moved to v.*
2. *Let q be the number of vertices with at least one pebble. If there are all together more than $2(2^{n(k-1)+1}) - q$ pebbles, then two pebbles can be moved to v.*

Proof. We induct on n. It is trivially true for $n = 1$.

Suppose it is true for $n' < n$. Let $v \in C_t, 1 \leq t \leq n$, be any specified vertex of ZZ_n. We will first show (1) holds. We consider the following cases.

Case (1.1): $1 < t < n$.

Then the graph can be partitioned into two subgraphs of ZZ_n, say S_1 and S_2, where $S_1 \cong ZZ_t$ and $S_2 \cong ZZ_{s(=n-t)}$. Clearly

$v \in S_1$ and $p(v) = 0$ by Remark 3.1. If at least $2^{t(k-1)+1}$ pebbles are distributed on S_1, then by induction one pebble can be moved to v. Therefore we may assume S_1 contains at most $2^{t(k-1)} - 1$ pebbles. By Observation 2.4 we have remaining at least $2^{(s+1)(k-1)+1}$ pebbles are distributed on $S_2 - \{e\}$, where $e \in E(S_1 \cap S_2)$. Again by induction, we can move a pebble to v by using that remaining number of pebbles on $S_2 \cup C_t$.

Case (1.2): $t = 1 \; or \; n$.

Let us take $v \in C_n$. Now the graph can be partitioned into two subgraphs of ZZ_n, say S_1 and S_2 where $S_1 \cong ZZ_{n-1}$ and $S_2 \cong ZZ_1 \cong C_n$. Clearly $v \in S_2$. We consider the following possibilities to prove Case (2).

Subcase 1.2(*a*): S_1 does not satisfy the $2 -$ pebbling property.

Let q_1 be the number of vertices in S_1 with at least one pebble. Consider S_1 contains at most $2(2^{(n-1)(k-1)+1}) - q_1$ pebbles.

Claim (1) S_2 has at least 2^k pebbles.

Therefore we have

$$2^{n(k-1)+1} - (2(2^{(n-1)(k-1)+1}) - q_1)$$
$$= 2^{(n-1)(k-1)+1}(2^{k-1} - 2) + q_1$$
$$= 2^{n(k-1)+1}\left(1 - \frac{2}{2^{k-1}}\right) + q_1$$

$$\geq \; 2(2^n + 2^{k-1}\left(1 - \frac{2}{2^{k-1}}\right) + q_1$$

$$= \; (2^{n-1} + 2^k)\left(1 - \frac{2}{2^{(k-1)}}\right) + q_1$$

$$\geq \; 2^k, \text{Since} k \; \geq 3.$$

Hence at least 2^k pebbles are distributed on S_2. Therefore we can move one pebble to v.

Subcase 1.2(*b*): S_1 satisfies the 2− pebbling property.

Assume S_2 contains at most $2^k - 1$ pebbles. Let $V\ (ZZ_n) = \{x, a_1, a_2, \ldots, a_{n(k-1)}, v, b_{n(k-1)}, \ldots, b_2, b_1\}$ and let v be our target vertex. Let $P_A = a_1a_2\ldots a_{n(k-1)}v$ and$P_B = b_1b_2\ldots b_{n(k-1)}v$. Note that $f(P_A) = f(P_B) = 2^{n(k-1)}$. Also $0 \leq q_A, q_B \leq n(k-1) \leq 2^{n(k-1)-2}$. Let p_x be the number of pebbles located on x. Define p_{a_i} and p_{b_i} similarly. Let W_A and W_B be the number of pebbles assigned on the vertices of P_A and P_B respectively. If either P_A or P_B has at least $2^{n(k-1)}$ pebbles then we can reach the vertex v. Therefore assume that $W_A < 2^{n(k-1)}$ and$W_B < 2^{n(k-1)}$. Then remaining pebbles are distributed on the vertex x. Without loss of generality, we may assume $W_A \geq W_B$.

Claim(2): $\frac{p_x}{2} + W_A \geq 2^{n(k-1)}$

For suppose not, we have $\frac{p_x}{2} + W_A < 2^{n(k-1)} - 1$.

$W_A < 2^{n(k-1)} - 1 - \frac{p_x}{2}$.

Since $W_B \leq W_A$,

$$W_A + W_B < \left(2^{n(k-1)} - 1 - \frac{p_x}{2}\right) + \left(2^{n(k-1)} - 1 - \frac{p_x}{2}\right)$$

$$= 2^{n(k-1)+1} - 2 - p_x$$

$$W_A + W_B + p_x < 2^{n(k-1)+1} - 2.$$

This contradicts the total number of pebbles distributed on the vertices of graph. Hence by claim (2), we can move a pebble to v. This establishes (1).

We will now prove (2). Suppose there are $2(2^{n(k-1)+1}) - q + 1$ pebbles assigned to the vertices of the graph ZZ_n. We have to show that two pebbles can be moved to v. Assume $p_v = 0$ by Remark 3.2. Let $v \in C_t, 1 \leq t \leq n$.

We consider the following cases.

Case (2.1): $1 < t < n$.

We partition the graph into two subgraphs, say S_1 and S_2, where $S_1 \cong ZZ_t$ and $S_2 \cong ZZ_s$. Here, $S_1 \cup S_2 \cong ZZ_n$, $S_1 \cap S_2 \cong C_t$ and $n = t + s - 1$. Define $q = q_{s_1} + q_{s_2} - q_t$ be the sum of occupied vertices of S_1, S_2 and C_t respectively. Let W_{S_1} and W_{S_2} be defined similarly as in above case. Suppose $W_{S_1} \geq 2(2^{t(k-1)+1}) - q_{s_1} + 1$, then we can move two pebbles to v.

Assume $W_{S_1} \leq 2(2^{t(k-1)+1}) - q_{S_1}$.

Claim (3): S_2 has at least $2(2^{s(k-1)+1}) - q_{S_2} + 1$ pebbles.

Now the remaining number of pebbles is at least $2(2^{n(k-1)+1}) - q + 1 - (2(2^{t(k-1)+1}) - q_{S_1})$ in S_2. In Observation 2.4, put $s = s - 1$

we get, $2^{(t+s-1)m} - 2^{tm} \geq 2^{(t+s-2)m}$

$\geq 2^{sm}$ since $t \geq 2$.

Therefore, by using above calculation we get,

$$2\left(2^{n(k-1)+1}\right) - q + 1 - \left(2\left(2^{t(k-1)+1}\right) - q_{S_1}\right)$$
$$\geq 2\left(2^{s(k-1)+1}\right) - q_{S_2} + 1$$

Hence we can move two pebbles to v.

Case (2.2): $t = 1$ or n.

Let us take $v \in C_n$. Let $P_A = a_1 a_2 \ldots . a_{n(k-1)} v$ and $P_B = b_1 b_2 \ldots b_{n(k-1)} v$. We have to move two pebbles to v. Suppose $W_A \geq 2(2^{n(k-1)}) - q_A + 1$ or $W_B \geq 2(2^{n(k-1)}) - q_B + 1$, then we can put two pebbles to v. Therefore we may assume $W_A \leq 2(2^{n(k-1)}) - q_A$ and $W_B \leq 2(2^{n(k-1)}) - q_B$. Suppose $W_A \geq 2(2^{n(k-1)-1}) - q_A + 1$ and $W_B \geq 2(2^{n(k-1)-1}) - q_B + 1$, then we are done. Therefore we assume that $W_A \geq 2(2^{n(k-1)-1}) - q_A + 1$ or $W_B \geq 2(2^{n(k-1)-1}) - q_B + 1$.

Now we may fix

$2(2^{n(k-1)}) - q_A \geq W_A \geq 2(2^{n(k-1)-1}) - q_A + 1$, since

$q_v = 0$, still we can move one pebble to v.

Claim (4): $p_x + W_B \geq 2(2^{n(k-1)}) - (q_B + q_x) + 1$.

$p_x + W_B \geq 2(2^{n(k-1)+1}) - q + 1 - (2(2^{n(k-1)}) - q_A)$

$= 2(2^{n(k-1)+1}) - 2(2^{n(k-1)}) - (q_B + q_x) + 1$

$= 2(2^{n(k-1)}) - (q_B + q_x) + 1$

Then we can move additional one pebble to v.

Now assume $W_A \leq 2(2^{n(k-1)-1}) - q_A$. Fix $W_A \geq 2^{n(k-1)-1}$, then one pebble can be moved to $a_{n(k-1)}$. Then

$$W_B + p_x \geq 2(2^{n(k-1)+1}) - q + 1 - (2(2^{n(k-1)-1}) - q_A)$$
$$= (2(2^{n(k-1)}) - q_B) + (2^{n(k-1)} - q_x + 1).$$

Since $W_B \leq W_A$, at least $2^{n(k-1)}$ pebbles are in x and using these pebbles in x we can move one pebble to $a_{n(k-1)}$. Recall that already we put one pebble on $a_{n(k-1)}$ by using the pebbles in P_A. Therefore we can put one pebble to v. Then remaining at least $2(2^{n(k-1)}) - (q_B + q_x) + 1$ pebbles are on $P_B \cup \{x\}$.

Thus we can move two pebbles to $b^{n(k-1)}$ and we are done.

Assume $W_A \leq 2^{n(k-1)-1} - 1$. Now we have to calculate the pebbles in $P_B \cup \{x\}$.

We have,

$W_B + p_x \geq 2(2^{n(k-1)+1}) - q + 1 - 2^{n(k-1)-1} + 1$

$\geq 2(2^{n(k-1)} - (q_A + q_B + q_x) + 1 + 2^{n(k-1)+1} -$
$2^{n(k-1)-1} + 1 - q_A$

$\geq 2\left(2^{n(k-1)}\right) - (q_B + q_x) + 1 + 2^{n(k-1)+1} -$
$2^{n(k-1)-1} + 1 - 2^{n(k-1)-2}$,
$W_B + p_x \geq 2\left(2^{n(k-1)}\right) - (q_B + q_x) + 1 + 2^{n(k-1)}$
$-2^{n(k-1)-2} + 1.$

Since $W_B \leq W_A \leq 2^{n(k-1)-1} - 1$. Therefore we have $W_B \leq 2^{n(k-1)-1} - 1$.

Then remaining $2^{n(k-1)+1} - q + 1 - 2(2^{n(k-1)-1} - 1)$ pebbles are in x. Now we can write $p_x = x_1 + x_2$, where x_1 and x_2 are any two non-negative integers. The reader can easily verify that, for any distribution belonging to this condition,
we have

$$x_1 + W_A \geq 2\left(2^{n(k-1)}\right) - (q_A + q_x) + 1 \; and$$

$$x_2 + W_B \geq 2(2^{n(k-1)}) - (q_B + q_x) + 1, when \; q_A, q_B \; and \; q_x \geq 1.$$

Therefore we can put two pebbles to v. Suppose any one of $q_A = 0$ or $q_B = 0$, then we can easily move two pebbles to v. Hence we are done.

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